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Error Estimates for Extrapolation Operators*

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1. INTRODUCTION

In numerical analysis often the following problem arises. Compute an approximation of the value $f(x)$ at a prescribed point $x \in I$ where for the real function f defined on the compact interval $I = [a, b]$ only the values $f(x_i)$ are assumed to be known, $x_0, ..., x_n \in I$. This problem usually will be solved by choosing a Čebyšev system $(f_0, ..., f_n)$ of real functions defined on I and computing the value $Hf(x)$ of the unique linear combination Hf of $f_0, ..., f_n$ that interpolates f at the knots $x_0, ..., x_n$. If $(f_0, ..., f_n)$ is a complete Čebyšev system on I (i.e., for $k = 0, ..., n$, $(f_0, ..., f_k)$ is a Čebyšev system on I) then $Hf(x)$ can be computed recursively either using generalized divided differences and Newton's interpolation formula [4] or using a generalization of the Neville–Aitken algorithm [5]. Naturally the question arises: How good is this approximation? For all complete Čebyšev systems $(f_0, ..., f_n)$ of continuous functions the interpolation or extrapolation error can be estimated in terms of the modulus of continuity of the function f/f_0 and in terms of the Lebesgue function of an interpolation operator naturally associated with H, which can itself be estimated in terms of the moduli of continuity of the functions f_i/f_0 $(i = 1, ..., n)$ and the "Čebyšev moduli" of the subsystems $(f_0, ..., f_i)$.

2. ERROR BOUNDS FOR EXTRAPOLATION OPERATORS

The following theorem is an extension of a result of Braß and Günttner [1] concerning interpolation of continuous functions by algebraic polynomials to the more general case of interpolation by functions which form a complete

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oriented Čebyšev system. Following Zielke [8], a complete Čebyšev system $(f_0, ..., f_n)$ on I will be called oriented if, for $k = 0, ..., n$,

$$V\binom{f_0, \ldots, f_k}{t_0, \ldots, t_k} := \det f_j(t_i)$$

has a constant sign for all $t_0, ..., t_k \in I$ with $t_0 < \cdots < t_k$. Clearly, this will be fulfilled if $f_0, ..., f_n$ are real continuous functions on I.

THEOREM 1. *Let f_0 be the constant* 1 *and let $(f_0, f_1, ..., f_n)$ be a complete oriented Čebyšev system on I. Let $(x_0, ..., x_n) \in I^{n+1}$ be a system of simple knots ordered in increasing order, and for an arbitrary real function f defined on I let Hf denote the unique linear combination of* $1, f_1, ..., f_n$ *interpolating f at $x_0, ..., x_n$. Then for all $x \in I$,*

$$|Hf(x) - f(x)| \leqslant \tfrac{1}{2}[\lambda(x) + 1]\,\omega(f; \Delta),$$

where $\Delta := \max\{x_0 - a, x_1 - x_0, x_2 - x_1, ..., x_n - x_{n-1}, b - x_n\}$, *$\omega$ is the modulus of continuity of the function f on I, and λ is the Lebesgue function of H:*

$$\lambda(x) := \sum_{i=0}^{n} |l_i(x)|, \qquad l_i(x) := V\binom{f_0, \ldots \ldots, f_n}{x_0, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n} \Big/ V\binom{f_0, \ldots, f_n}{x_0, \ldots, x_n}.$$

Proof. Following Braß and Günttner [1] it is sufficient to show that $h_k(x) := \sum_{m=0}^{k} l_m(x)$ changes sign for $x \geqslant x_k$ precisely at $x_{k+1}, ..., x_n$ and that $\bar{h}_k(x) := 1 - h_k(x)$ changes sign for $x \leqslant x_{k+1}$ precisely at $x_0, ..., x_k$ $(k = 0, ..., n - 1)$. But, observing that $h_k, \bar{h}_k \in \operatorname{span}\{1, f_1, ..., f_n\} \backslash \operatorname{span}\{1, f_1, ..., f_{n-1}\}$ for $k = 0, ..., n - 1$ (the divided differences of these functions with respect to the knots $x_0, ..., x_n$ and the system $(1, f_1, ..., f_n)$ are different from zero, as can be seen from the recurrence relation (1) below) this follows from a result of Zielke concerning the oscillation properties of such functions [8, Lemma 3, p. 174].

Remarks. 1. It is an open question whether Theorem 1 holds for arbitrary Čebyšev systems $(1, f_1, ..., f_n)$. Braß and Günttner have shown that Theorem 1 also holds for interpolation by trigonometric polynomials at the knots subdividing $[0, 2\pi]$ into congruent intervals.

2. The observation of O. Kis that the estimate in the classical cases of interpolation cannot be improved remains valid in the more general case. As noted by Braß and Günttner, this is an immediate consequence of the facts $\omega(f; \Delta) \leqslant 2\|f\|$ ($\|\cdot\|$ = sup-norm) and $\|H - I\| = \|H\| + 1$ (I = identity); the last equality was proved by Cheney and Price [2] in a much more general setting.

3. Estimates of the Lebesgue Functions

The Lebesgue function $\lambda(x)$ can be estimated by a repeated use of Theorem 1. The following definition will be needed.

Definition. *Let* $(f_0, \ldots, f_n)$ *be a complete Čebyšev system on* I. *For* $k = 0, 1, \ldots, n$, *the function*

$$k = 0: \qquad 0 < \delta \to \rho_0(\delta) := \inf\{ f_0(x) : x \in I\},$$

$$k \geq 1: \quad \left(0, \frac{b-a}{k}\right] \ni \delta \to \rho_k(\delta) := \inf\{ |[x_1, \ldots, x_k; f_k] - [x_0, \ldots, x_{k-1}; f_k]| : \\ a \leq x_0 < x_1 < \cdots < x_k \leq b, \\ \min_{1 \leqslant i \leqslant k} (x_i - x_{i-1}) \geq \delta\}$$

will be referred to as the Čebyšev modulus of the subsystem $(f_0, \ldots, f_k)$ *on* I. *Here*

$$[y_0, \ldots, y_j; f] := V\binom{f_0, \ldots, f_{j-1}, f}{y_0, \ldots, y_{j-1}, y_j} \Big/ V\binom{f_0, \ldots, f_{j-1}, f_j}{y_0, \ldots, y_{j-1}, y_j}$$

denotes the divided difference of f *with respect to the Čebyšev system* $(f_0, \ldots, f_j)$ *and the simple knots* $y_0, \ldots, y_j$.

Example. For the Čebyšev system of the power functions $(1, x, x^2, \ldots, x^n)$ it is well known that $\rho_k(\delta) = k\delta$, for $k \geqslant 1$.

Remark. For an extended complete Čebyšev system $(f_0, \ldots, f_n)$ $(n \geq 1)$ on $I = [a, b]$, with $f_i^{(p)}(a) = 0$, $p = 0, 1, \ldots, i-1$; $i = 1, 2, \ldots, n$, one has [3, Theorem 1.2, p. 379]:

$$f_0(x) = w_0(x),$$

$$f_1(x) = w_0(x) \int_a^x w_1(t_1)\, dt_1,$$

$$\ldots$$

$$f_n(x) = w_0(x) \int_a^x w_1(t_1) \int_a^{t_1} w_2(t_2) \int_a^{t_2} \cdots \int_a^{t_{n-1}} w_n(t_n)\, dt_n \cdots dt_1,$$

where $w_i \in C^{n-i}[a, b]$ are strictly positive functions. It is possible to estimate the Čebyšev moduli of the subsystems $(f_0, f_1, \ldots, f_i)$ on I from below [7]. If, for $i = 0, \ldots, n$, $m_i := \min\{w_i(x) : x \in I\}$, $M_i := \max\{w_i(x) : x \in I\}$, then for all permissible $\delta > 0$ and $k = 1, \ldots, n$,

$$\rho_k(\delta) \geqslant \delta m_k \prod_{i=1}^{k-1} \left(\frac{m_i}{M_i}\right)^{2(k-i)}$$

and

$$\rho_k(\delta) \geqslant \frac{\delta m_k}{(k-1)!} \prod_{i=1}^{k-1} \left(\frac{m_i}{M_i}\right)^{k-1-i}.$$

For $\delta \to 0$ the first estimate cannot, in general, be improved: It is easily seen that, for the system of power functions, equality holds.

The divided differences can be computed and thus also estimated by the recurrence [6]:

$$(1) \qquad [x_0, \dots, x_n \mid f] = \frac{[x_1, \dots, x_n \mid f] - [x_0, \dots, x_{n-1} \mid f]}{[x_1, \dots, x_n \mid f_n] - [x_0, \dots, x_{n-1} \mid f_n]}.$$

It is easily seen by induction that if $\min_{1 \leqslant j \leqslant n}(x_j - x_{j-1}) \geqslant \delta > 0$, then

$$(2) \qquad |[x_0, \dots, x_n \mid f]| \leqslant \frac{2^n M}{\prod_{i=0}^{n} \rho_i(\delta)}$$

where $M := \sup\{|f(x_i)|: i = 0, \dots, n\}$. In general this estimate cannot be improved, for the system of power functions, for equidistant knots, and with $f(x_i) = (-1)^i M$, there is equality in (2).

THEOREM 2. *Let $(1, f_1, \dots, f_n)$ $(n \geqslant 1)$ be a complete oriented Čebyšev system on $I = [a, b]$ and let ρ_k be the Čebyšev moduli of its subsystems. Let $(x_0, \dots, x_n) \in I^{n+1}$ be a system of simple knots ordered in increasing order. Then for all $x \in I$*

$$\tfrac{1}{2}(\lambda(x) + 1) \leqslant \prod_{i=1}^{n} \left[1 + \frac{2^{i-1}\omega(f_i\,;\Delta)}{\prod_{k=1}^{i} \rho_k(\delta)}\right]$$

where $\delta := \min_{1 \leqslant j \leqslant n}(x_j - x_{j-1})$ and Δ and λ are defined in Theorem 1.

Proof. For a fixed system of knots $x_0 < x_1 < \cdots < x_n$, define a function

$$\varphi_i \in \operatorname{span}\{1, f_1, \dots, f_n\}$$

by

$$\varphi_i(x_j) := \operatorname{sign} l_j(x) \ (j = 0, \dots, n), \qquad \text{for} \quad x_{i-1} < x < x_i \ (i = 1, \dots, n)$$

and let φ_0 and φ_{n+1} be defined likewise for $a \leqslant x < x_0$ or $x_n < x \leqslant b$, respectively. If $(x_{\nu_0}, \dots, x_{\nu_n})$ is any permutation of $(x_0, \dots, x_n)$, then for $x \in I$ different from each knot, we have, by Newton's interpolation formula, with i suitably chosen:

$$\begin{aligned} \lambda(x) &= \sum_{j=0}^{n} |l_j(x)| = H\varphi_i(x) \\ &= \varphi_i(x_{\nu_0}) + \sum_{k=1}^{n} [x_{\nu_0}, \dots, x_{\nu_k} \mid \varphi_i] \cdot \{f_k(x) - H_{\nu_0, \nu_{k-1}} f_k(x)\} \end{aligned}$$

where $H_{\nu_0, \ldots, \nu_{k-1}} f$ is the unique linear combination of $1, f_1, \ldots, f_{k-1}$ that interpolates to f at the knots $x_{\nu_0}, \ldots, x_{\nu_{k-1}}$. For a given $x \in I$ distinct from each knot, choose the first knot x_{ν_0} to be the nearest to x among all knots $x_0, \ldots, x_n$; then choose x_{ν_1} to be the nearest knot to x among the remaining knots, and so on. Summarized, depending on x there can be chosen a permutation $(x_{\nu_0}, \ldots, x_{\nu_n})$ of the knots $(x_0, \ldots, x_n)$ such that $\lambda_{\nu_0, \nu_0}(x) = 1$ and

$$\lambda(x) \leq \lambda_{\nu_0, \ldots, \nu_{n-1}}(x) + \tfrac{1}{2}(\lambda_{\nu_0, \ldots, \nu_{n-1}}(x) + 1) \cdot \omega(f_n ; \Delta) \cdot |[x_0, \ldots, x_n ; \varphi_i]|$$

where $\lambda_{\nu_0, \ldots, \nu_{n-1}}$ is the Lebesgue function of $H_{\nu_0, \ldots, \nu_{n-1}}$. Here Theorem 1 is applied to f_n and to a smaller interval contained in I such that the maximal distance of adjacent knots among $x_{\nu_0}, \ldots, x_{\nu_{n-1}}$ can be replaced by Δ during our choice of these knots. An application of estimate (2) to the present case and an induction argument yield the estimate stated in Theorem 2.

Both Theorems 1 and 2 are easily extended to complete Čebyšev systems $(f_0, \ldots, f_n)$ where the first function f_0 is not constant by considering the system $(1, f_1/f_0, \ldots, f_n/f_0)$. Using the same notation as above with the Čebyšev moduli now defined with respect to $(f_0, \ldots, f_n)$, it is easily seen that the following result holds.

COROLLARY. *If $(f_0, \ldots, f_n)$ is a complete oriented Čebyšev system on I, then for all $x \in I$,*

$$|Hf(x) - f(x)| \leq \frac{|f_0(x)|}{2} \, \omega(f/f_0 ; \Delta) \prod_{i=1}^{n} \left[1 + \frac{2^{i-1} \omega(f_i/f_0 ; \Delta)}{\prod_{k=1}^{i} \rho_k(\delta)}\right].$$

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